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Non-proliferation of pre-images in integrable mappings

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Abstract. We present an integrability criterion for rational mappings based on two requirements. First, that a given point should have a unique pre-image under the mapping and, second, that the spontaneously appearing singularities be confined to a few iteration steps. We present several examples of known integrable mappings that meet these requirements and, also, use our algorithm in order to derive new examples of integrable mappings.

1. Introduction

Integrability detectors are rare even for continuous systems. Recent progress in the integrability of discrete systems has spurred activity in this direction, leading to interesting results of great variety. In order to characterize discrete integrability, Arnold has introduced and investigated the concept of complexity for mappings in a plane [1]. Arnold defines the complexity from the number of intersection points of a fixed curve with the image of a second given curve under the kth iteration of the mapping. For a polynomial mapping the growth of the number of intersection points is in general exponential in k. However, for integrable mappings the growth is only polynomial in k. This result is included in a more general analysis presented by Veselov [2] discussing the dynamics of multiple-valued mappings (correspondences) and the growth of the number of different images (and pre-images). Veselov, too, has linked integrability to slow growth.

Our approach of discrete integrability is different from the above since it was based, essentially, on rational (rather than polynomial) mappings. For rational mappings, an important question is what happens whenever accidentally (i.e. depending on the initial conditions) a denominator vanishes, leading to a divergent mapping variable. In general one expects this singularity to propagate indefinitely under the mapping iterations, but it turns out that for *integrable* mappings these singularities disappear after a few steps. This observation has led to the proposal of the singularity confinement [3] criterion as a detector of discrete integrability. Its efficiency has already been proven through the derivation of new integrable systems leading to the discovery of discrete Painlevé equations [4].

It appears that the two notions of *slow growth* and *confined singularities* play an important role in the characterization of the integrable discrete systems. In what follows, we will try to present our approach which is based on both notions. We will, first, introduce the notion of pre-image non-proliferation as well as the algorithm for its assessment. Based on the slow-growth principle, we claim that the number of pre-images of a given point should not grow exponentially fast, which, when we consider mappings rather than general correspondences, can only mean a single pre-image. Therefore, for the practical

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implementation we will present of this criterion, we will require that the inverse of the mapping be uniquely defined. The singularity confinement conjecture will also be extended in the following sections, essentially through the extension of the notion of singularity of a mapping. Apart from the singularity related to a divergence, we will consider as appearance of a singularity all the instances where the mapping accidentally loses some degrees of freedom. (The precise mechanism will become clear in section 3.) Confinement of this singularity consists in the recovery of these lost degrees of freedom usually through the appearance of an indeterminate form like 0/0. While pre-image non-proliferation is only a necessary condition for integrability, we conjecture that its combination with singularity confinement leads to a sufficient condition for integrability of discrete systems.

In what follows, we will limit ourselves to rational explicit mappings, i.e.

$$x_i' = f_i(x_1, x_2, \ldots x_N) \qquad i = 1, \ldots N \tag{1}$$

with rational f_i's and where the x_i's are, in general, complex numbers. Integrability in our sense means one of the following things.

(a) Existence of a sufficient number of rational $\Phi_k(x_1, \ldots x_N) = C_k$, the values of which are invariant under the action of the mapping.
(b) Linearizability of the mapping through a Cole–Hopf-type transformation $x_i = P_i/Q_i$ whereupon the mapping reduces to a linear one for the P_i's, Q_i's.
(c) Linearizability through a Lax pair. In this case, the mapping is the compatibility condition of a linear system of differential-difference, q-difference or pure difference equations.

The above are not definitions but rather illustrations of the various types of integrability. It may well occur, as in the case of Quispel's mappings [5], that the existence of one invariant reduces the mapping to a correspondence of the form $F(x, x') = 0$ that can be parametrized in terms of elliptic functions. In other cases, integration using the rational invariants may lead to a transcendental equation like the discrete Painlevé ones. All of the above types of integrability have been encountered in the discrete systems that we have studied [6, 7]. The reason for the above classification is to emphasize the parallel existing between the continuous and discrete cases. In the next section, we examine specific examples of mappings and along the way formulate our conjecture on pre-image non-proliferation.

2. Examples of integrable mappings and the pre-image non-proliferation criterion

Let us start with a very simple example of rational mapping, in which the growth of the number of pre-images must be invoked. In [6], we studied the one-component, two-points mapping of the form

$$x' = f(x) \tag{2}$$

where f is rational. Singularity confinement considerations lead to

$$f(x) = \alpha + \sum_k \frac{1}{(x - \beta_k)^{\nu_k}} \tag{3}$$

with positive integer ν_k, provided that for all k, $\beta_k \neq \alpha$. Indeed, if $x = \beta_k$ at some step, then x' diverges, $x'' = \alpha$ and x''' is finite. So the mapping propagates without any further difficulty. However, if we consider the backward evolution, then (2) solved for x in terms of x' leads to multi-determinacy and the number of pre-images grows exponentially with the number of backward iterations. Indeed, the only mapping of the form (2)–(3) with no growth is just the homographic

$$x' = \frac{ax+b}{cx+d} \tag{4}$$

which is the discrete form of the Riccati equation [8]. Thus, in this case, the argument of slow growth of the number of pre-images of x is essential in deriving the form of the discrete Riccati equation.

Another classical example in the domain of integrable mappings is the Quispel family. In [5], Quispel and collaborators have shown that the mappings

$$x' = \frac{f_1(y) - f_2(y)x}{f_2(y) - f_3(y)x} \qquad y' = \frac{g_1(x') - g_2(x')y}{g_2(x') - g_3(x')y} \tag{5}$$

are integrable, provided the f_i's, g_i's are specific quartic polynomials involving 18 parameters. We remark here that the mapping is staggered, i.e. while x' is defined in terms of (x, y), y' is defined in terms of (x', y). It is precisely this staggered structure that allows one to define a unique pre-image to (x, y). As Quispel has shown in [9], the mapping (5) is reversible, which means that it can be written as a product of two involutions. It is not clear whether reversibility is a prerequisiste for integrability, but, though reversibility ensures that the pre-image is unique, there still exist reversible systems that are not integrable.

Reversible integrable mappings have also been considered by the Paris group [10] in their works based on the study of lattice spin and vertex models. They have shown that the transformations involved are in fact symmetries of the Yang–Baxter equations. These symmetries are constructed as the product of a pair of non-commuting involutions: thus the mapping is reversible and generically of infinite order. Still, it is interesting to investigate the mechanism for the non-proliferation of pre-images in this case and present the algorithm that one should use. Let us illustrate this in the case of the mapping

$$x' = \frac{x + y - 2xy^2}{y(y-x)} \qquad y' = \frac{x + y - 2yx^2}{x(x-y)}. \tag{6}$$

The first step consists of considering the system of N equations $x_i' - f_i(x_k) = 0$ (here $N = 2$ and $x = x_1$, $y = x_2$) and successively eliminating all but one of the x_k's. In the case of the mapping (6) the resultant in x after eliminating y (or vice-versa) is a fifth-degree polynomial in x (resp y) with coefficients depending on x' and y'. Next we factorize this resultant. Pre-image non-proliferation requires that only *one* factor depend on x', y', the other factors being associated with indeterminate forms 0/0. We find in this particular example the factors $x^2(x^2-1)$ (resp $y^2(y^2-1)$) and one last factor leading to

$$x = \frac{x' - y'}{y'^2 + x'y' - 2} \qquad y = \frac{y' - x'}{x'^2 + x'y' - 2}. \tag{7}$$

This is the typical situation for integrable rational mappings. The factorization of the resultant gives the unique inverse of the mapping along with particular values (here $x = y = 0, \pm 1$) corresponding to the indeterminate forms of the mappings.

Slow-growth arguments have been used by Veselov in his studies of the integrability of mappings and correspondences. In particular, he has studied in [2] the integrability of polynomial mappings and has shown that the mapping $x' = P(x, y), y' = Q(x, y)$ is integrable (in the sense that it possesses a non-constant polynomial integral $\Phi(x, y)$) if there exists a polynomial change of coordinate variables transforming the mapping to triangular form:

$$x' = \alpha x + P(y) \qquad y' = \beta y + \gamma \tag{8}$$

for polynomial P. Moreover, he has shown that in this case the complexity of the mapping is bounded. The important feature in (8) is the fact that the equation for y' is linear. Thus the inversion of (8) is straightforward. Thanks to the triangular form, the integration of (8) is reduced to the solution of two affine mappings, first for y and then for x.

One more interesting illustration of the pre-image non-proliferation algorithm is provided by the discrete Painlevé equations that we derived in [11] and which are *not* of Quispel form. We have found there that the mappings

$$x' = \frac{xy(a(y+1) - xy^2)}{a(y+1)^2} \qquad y' = \frac{a(y+1)(xy^2 - (y+1)(a - zy)}{(a(y+1) - xy^2)^2} \tag{9}$$

where a = constant and z is linear in the lattice variable, represent a discrete form of the P_{I} equation. In order to check the pre-image non-proliferation, we eliminate x (or y) from (9) and factorize the resultant. We find, as expected, factors related to exceptional points $x = 0$, $y = 0$, $y = -1$, and a unique inverse that reads

$$x = \frac{x'(x'y' + z)[(x'y' + z)^2 + a(y' + 1)(z - x')]}{a(y' + 1)^2} \qquad y = \frac{a(z - x')(y' + 1)}{(x'y' + z)^2}. \tag{10}$$

Thus in this example, too, as in all previous ones, integrability is related to non-growth of the number of pre-images.

3. Extending the singularity confinement criterion

In the previous section, we encountered several examples of integrable mappings, all of which satisfied the no-growth property. Here, we will apply the pre-image non-proliferation criterion in order to construct explicitely integrable mappings. However, since this criterion furnishes only a necessary condition for integrability, we will supplement it by singularity confinement, our conjecture being that the combination of the two criteria is sufficient for discrete integrability.

Applying the pre-image non-proliferation algorithm to a general mapping can easily lead to untractable calculations. If, however, there are not too many free parameters in the mapping the implementation of the criterion is straightforward. In what follows, we will limit ourselves to simple two-component, two-point mappings of the form

$$x' = \frac{Q_1(x, y)}{Q(x, y)} \qquad y' = \frac{Q_2(x, y)}{Q(x, y)} \tag{11}$$

where Q, Q_1 and Q_2 are quadratic polynomials in x and y. By applying a general linear transformation on this mapping we can reduce the (common) denominator to one of the two canonical forms $Q = xy - 1$ or $x^2 - y$ (or any of the degenerate forms $Q = xy$, $x^2 - 1$ or x^2). Let us start with the mapping h:

$$x' = \frac{a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}}{xy - 1} \tag{12a}$$

$$y' = \frac{b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{10}x + b_{01}y + b_{00}}{xy - 1}. \tag{12b}$$

One assumption that we will introduce here is $a_{02} = 0$ since it leads to a great simplification of the calculations. As a first step in the pre-image non-proliferation algorithm we eliminate y between (12a) and (12b) for given x' and y' and obtain a resultant that is a fourth-degree polynomial in x. We demand that three of the roots be independent of x', y' and denote them by x_1, x_2 and x_3. We then demand that whenever $x = x_i$, $y = 1/x_i$, $i = 1, 2, 3$, both x' and y' have the indeterminate form 0/0. Calling

$$\Sigma = x_1 + x_2 + x_3 \qquad P = x_1x_2 + x_2x_3 + x_3x_1 \qquad \Pi = x_1x_2x_3 \tag{13}$$

we find a unique pre-image if and only if

$$\begin{aligned} &a_{10} = -a_{20}\Sigma \qquad a_{01} = -a_{20}\Pi \qquad a_{00} = a_{20}P - a_{11} \\ &b_{10} = -b_{20}\Sigma - b_{02}/\Pi \qquad b_{01} = -b_{20}\Pi - b_{02}P/\Pi \\ &b_{00} = b_{20}P - b_{11} + b_{02}\Sigma/\Pi\,. \end{aligned} \tag{14}$$

With (14) the mapping satisfies the pre-image non-proliferation requirement. This is *not*, however, sufficient for integrability. What we must also demand is that the mapping have confined singularities. The simplest kind of singularity is whenever the denominator vanishes, i.e. $Q(x, y) = 0$ or, in our example, $y = 1/x$. However, since in the present case the numerators Q_1, Q_2 are also quadratic, the singularity is confined in one step: $Q(x, y) = 0$ leads to diverging x', y' and, because the degrees of numerators and (common) denominator are equal, this leads to finite x'' and y''. So the study of this singularity does not introduce any constraint on the mapping. But the vanishing of the denominators is *not* the only singularity of the mapping: a subtler singularity may exist.

Normally for a general N-component mapping, N free parameters, introduced by the initial conditions, must be present at every step. Now, it may happen that at some iteration one (or more) degress of freedom be lost. The condition for this to occur is that the Jacobian of $(x_1', x_2', \ldots x_N')$ with respect to $(x_1, x_2, \ldots x_N)$ vanishes. For a general mapping $x_i' = f_i(x_k)$ this reads

$$J = \begin{vmatrix} \partial x_1'/\partial x_1 & \partial x_1'/\partial x_2 & \ldots & \partial x_1'/\partial x_N \\ \partial x_2'/\partial x_1 & \partial x_2'/\partial x_2 & \ldots & \partial x_2'/\partial x_N \\ \vdots & \vdots & \ddots & \vdots \\ \partial x_N'/\partial x_1 & \partial x_N'/\partial x_2 & \ldots & \partial x_N'/\partial x_N \end{vmatrix} = 0\,. \tag{15}$$

How can this singularity be confined? By this we mean that the mapping must recover the lost degree of freedom. For a rational mapping of the kind we are considering, this can be realized if some of the mapping's variables assume an indeterminate form 0/0. In that case new free parameters can be introduced and the mapping recovers its full dimensionality.

Let us apply this criterion to the mapping h defined by (12a), (12b). The Jacobian readily factorizes and we obtain three factors, leading to either of the equations:

$$x + x_i - \Sigma + y\Pi/x_i = 0 \qquad i = 1, 2, 3\,. \tag{16}$$

Thus whenever (16) is satisfied a singularity appears (in the sense of the loss of one degree of freedom). For the confinement of this singularity (at the x'', y'' level) we must have $x'y' - 1 = 0$ and the numerators of both x'' and y'' must vanish. First we supplement the condition for the vanishing denominator $x'y' - 1$. This leads to a number of equations that, in fact, fully specify the remaining a, b coefficients and moreover put a constraint on x_1, x_2, x_3. The latter can be written (up to an odd permutation of x_1, x_2, x_3) as

$$3x_1x_2x_3 = x_1^2x_3 + x_2^2x_1 + x_3^2x_2\,. \tag{17}$$

For the a, b we obtain

$$a_{11}^3 = -\Pi \qquad a_{20} = -\frac{1}{a_{11}} \qquad b_{02} = a_{11} \qquad b_{20} = \frac{1}{\Pi} \qquad b_{11} = \frac{P}{\Pi} - \frac{1}{a_{11}}\,. \tag{18}$$

In fact, the simplest way to parametrize equations (14), (17) and (18) is to take $a_{11} \equiv a$ as a basic parameter. Introducing one further parameter, ω, we can express x_1, x_2, x_3 as: $x_1 = -a\omega$, $x_2 = a(1+1/\omega)$ and $x_3 = a/(1+\omega)$. It turns out that once conditions (14), (17) and (18) are implemented, the numerators of both x'' and y'' automatically vanish. Thus the mapping h is singularity confining and, according to our conjecture, it should be integrable. This is indeed the case and one invariant can be found. It reads:

$$\prod_{i=1}^{3} \frac{x + x_i - \Sigma + y\Pi/x_i}{x - x_i} = (-1)^n K \tag{19}$$

i.e. the product on the LHS, instead of being strictly constant, alternates in sign between even and odd iterations. One should, in principle, take the square of the LHS in order to find a true constant. A closer inspection of the mapping (motivated by the form of the invariant) reveals an even simpler structure: the mapping is periodic with period 6, i.e. $h^6 = I$.

Finally, the mapping can be cast in a much simpler form if one uses the scaling freedom in order to reduce the number of the parameters from two to one. We shall not enter into these details but just give the final result:

$$x' = \frac{-x^2 + xy + \sigma x - y + 2 - \sigma}{xy - 1} \tag{20a}$$

$$y' = \frac{-x^2 + (2-\sigma)xy + y^2 + (1+\sigma)x + (\sigma - 4)y + 1 - \sigma}{xy - 1}\,. \tag{20b}$$

In an analogous way we can treat the mapping p:

$$x' = \frac{a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}}{x^2 - y} \tag{21a}$$

$$y' = \frac{b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{10}x + b_{01}y + b_{00}}{x^2 - y} \tag{21b}$$

with $a_{02} = 0$. As in the previous case, we can ask that the resultant of the elimination of y between (21a) and (21b) for given x', y' (which is quartic in x), possess three roots independent on x', y', namely x_1, x_2 and x_3, corresponding to 0/0 indeterminacies. In a second step, the singularity confinement can be implemented in a perfect parallel to the case of the mapping h, leading to:

$$x' = \frac{x^2\Sigma - xy - xP + \Pi}{x^2 - y} \tag{22a}$$

$$y' = \frac{x^2\Sigma^2 - 2xy\Sigma + y^2 - (\Sigma P + \Pi)x + yP + \Sigma\Pi}{x^2 - y} \tag{22b}$$

with Σ, P, Π given by (13). The freedom of transformations ($x \to x+a$, $y \to y+2ax+a^2$) can be used in order to simplify this mapping. Finally, only one parameter remains and the mapping reads

$$x' = \frac{x(x - y - \rho)}{x^2 - y} \tag{23a}$$

$$y' = \frac{(x - y)(x - y - \rho)}{x^2 - y}. \tag{23b}$$

This mapping is indeed integrable, but in a trivial way: it is just an involution, $p^2 = I$. Still, the important point here is that the conjecture concerning the integrability of mappings that have non-proliferating pre-images and confined singularities is once again satisfied.

4. Conclusion

In the preceding sections, we have presented in detail the pre-image non-proliferation criterion, which, we conjecture, is a necessary condition for the integrability of rational mappings. Based on the slow-growth principle, this criterion consists in requiring that, in order to be a candidate for integrability, a rational mapping possesses a unique inverse. Thus, the number of pre-images of a given point through the mapping does not grow with the number of iterations (while an exponential increase is, generically, expected). Since this criterion offers only necessary conditions it cannot predict integrability but can be used as a fast 'screening' procedure. The successful candidates can then be tested for singularity confinement, which is more stringent but more difficult to implement. We conjecture the combination of the two criteria (pre-image non-proliferation and singularity confinement) to be an integrability predictor of the same efficiency as the Painlevé method for continuous systems. Several results already exist based on this approach and we expect the extension of the singularity confinement presented here to further widen its range of applications.

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